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1 Introduction

During this contract period, our work has focused on improvements to elliptic grid generation methods. There are two principle objectives in this project. One objective is to make the elliptic methods more reliable and efficient, and the other is to construct a modular code that can be incorporated into the National Grid Project (NGP), or any other grid generation code. Progress has been made in meeting both of these objectives. The two objectives are actually complementary. As the code development for the NGP progresses, we see many areas where improvements in algorithms can be made.

2 Current Research

During the past year, most of our research has been on the development of the elliptic method for generating surface grids. This has included the derivation of the orthogonality and control function options that are normally used with two and three-dimensional elliptic methods. These options were not available in the EAGLE grid generation code and, as far as is known, had not been discussed in any previous papers on elliptic surface grid generation. The convergence problems which were initially encountered with the surface code have been reduced by using a new form of upwind differencing on the elliptic equations and by reducing the size of the NURBS control net when computing the surface derivatives. However, convergence can still be a problem if the surface is poorly parameterized. Much of the success in the development of surface grid generation methodology has been due to the use of the standard NURBS representation of surfaces. The present state of development will be present at the International Conference on Hydro-Science and Engineering in Washington, DC. A copy of the paper "Elliptic Grid Generation on Surfaces", which will appear in the Proceedings, has been attached to this report.

The major task in the development of the elliptic volume code has been removing the dependence on the EAGLE data structure and writing a driver routine that can accept and process arbitrary grid blocks. At the same time we have improved some of the algorithms dealing with control functions and boundary orthogonality. There were several places where iterative procedures could be replaced by direct computations without a noticeable decrease in quality of the resulting grid.

The main elliptic volume and surface grid generation subroutines have been incorporated into the NGP system and are presently operational. The subroutines for the various control function and orthogonality options are also in the system, but they cannot be used since the panels for picking the options have not been built into the graphical user interface. However, the basic structure of the subroutines have been finalized so that code development can continue.

To be presented at the International Conference on Hydro-Science and Engineering, Washington, DC, June 1993.

ELLIPTIC GRID GENERATION ON SURFACES

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ABSTRACT

Elliptic grid generation methods for the construction of surface grids are discussed. New techniques for computing the control functions that determine the grid point distribution and for imposing boundary orthogonality are developed. Sample grids are presented to demonstrate the application of these methods in the solution of some common problems in grid generation.

INTRODUCTION

Grid generation is a critical area for many numerical simulation problems. Whether using structured or unstructured grids, it is up to the numerical analyst to distribute points throughout an often complex physical region. The distribution of the grid points and the geometric properties of the grid such as skewness, smoothness, and cell aspect ratios have a major impact on the accuracy of the simulation.

Most grid generation proceeds in stages. In two dimensions, the grid is first constructed on the boundary curve or curves and then constructed on the interior of the physical region. In three dimensions, the grid generation process proceeds from curves to surfaces and then to the interior of the physical region. At each stage of the grid generation process, the construction of the desired grid often follows in two steps. First, a grid is constructed by interpolation from the boundary of the region or surface, and then this grid is smoothed, and possibly modified in other ways, by an iterative procedure. The most successful iterative smoothing schemes are based on elliptic systems of partial differential equations that relate the physical and computational variables. The elliptic system may be applied to the boundary grids, the interior grids, or both. The elliptic system may preserve the original distribution of grid points or redistribute points as when constructing an adaptive grid. Orthogonality of the grid may be imposed along certain boundary components of the physical region.

The most difficult and less developed area of elliptic grid generation is that of grid generation on surfaces. This is because the grid must not only be smoothed, but the grid points must also stay on the surface. For surfaces defined by parametric equations, the simplest technique for achieving this goal is to work in parameter space rather than in the physical variables of the surface. There are some disadvantages associated with this approach. The differential equations become more complicated and contain two sets of

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derivatives, the derivatives of the physical variables with respect to the parametric variables and the derivatives of the parametric variables with respect to the computational variables.

This report will discuss some of the current problems in elliptic grid generation on surfaces and describe some new features and enhancements that have been recently developed. Grid orthogonality has been one area that has been of particular interest. Although orthogonal grids are generally not necessary for most numerical algorithms, there must be some limit on the skewness of the grid. Boundary orthogonality is also desireable whenever it is necessary to impose Neumann type boundary conditions on one or more of the physical variables in the problem. Also, some of the popular ways of treating Neumann boundary conditions are not accurate when the grid is not orthogonal and others become unstable if the grid is extremely skewed near the boundary.

ELLIPTIC SURFACE EQUATIONS

The elliptic equations for surface grids are a generalization of Laplace equations for harmonic mappings of plane regions. Let S be a simply-connected surface defined by the parametric equations

$$\vec{r} = \vec{r}(u, v), \quad 0 \le u, v \le 1.$$

Further, let u and v be functions of the computational variables ξ and η where $0 \le \xi, \eta \le 1$. Now a cartesian coordinate system in the computational rectangle generates a curvilinear coordinate system in the parametric rectangle which maps to a curvilinear coordinate system on the surface. Thus a uniform grid in the computational rectangle generates a curvilinear grid on the surface. The elliptic system of equations which relates the parametric and computational variables is given as

$$g_{22}(u_{\xi\xi} + Pu_\xi) - 2g_{12}u_{\xi\eta} + g_{11}(u_{\eta\eta} + Qu_\eta) = J^2\Delta_2 u \qquad (1)$$

$$g_{22}(v_{\xi\xi} + Pv_\xi) - 2g_{12}v_{\xi\eta} + g_{11}(v_{\eta\eta} + Qv_\eta) = J^2\Delta_2 v \qquad (2)$$

where

$$\begin{aligned}
g_{11} &= \overline{g}_{11}u_\xi^2 + 2\overline{g}_{12}u_\xi v_\xi + \overline{g}_{22}v_\xi^2 \\
g_{12} &= \overline{g}_{11}u_\xi u_\eta + \overline{g}_{12}(u_\xi v_\eta + u_\eta v_\xi) + \overline{g}_{22}v_\xi v_\eta \\
g_{22} &= \overline{g}_{11}u_\eta^2 + 2\overline{g}_{12}u_\eta v_\eta + \overline{g}_{22}v_\eta^2 \\
\Delta_2 u &= \overline{J}\Big[\frac{\partial}{\partial u}(\frac{\overline{g}_{22}}{\overline{J}}) - \frac{\partial}{\partial v}(\frac{\overline{g}_{12}}{\overline{J}})\Big] \\
\Delta_2 v &= \overline{J}\Big[\frac{\partial}{\partial v}(\frac{\overline{g}_{11}}{\overline{J}}) - \frac{\partial}{\partial u}(\frac{\overline{g}_{12}}{\overline{J}})\Big]
\end{aligned}$$

and

$$\overline{g}_{11} = \vec{r}_u \cdot \vec{r}_u, \quad \overline{g}_{12} = \vec{r}_u \cdot \vec{r}_v, \quad \overline{g}_{22} = \vec{r}_v \cdot \vec{r}_v$$

$$J = u_\xi v_\eta - u_\eta v_\xi, \quad \overline{J} = \sqrt{\overline{g}_{11}\overline{g}_{22} - \overline{g}_{12}^2}.$$

Although we will not attempt to derive these equations, they are related to conformal mappings onto surfaces and a discussion may be found in Mastin and Thompson (1984). Also, some of the theoretical results from harmonic mappings of plane regions can be verified for solutions of the above system. These same equations may also be derived from a differential geometric point of view as in the paper by Warsi (1990). The functions P and Q are used to control the grid point distribution and would not appear in the derivation from conformal or harmonic mappings.

The system of partial differential equations may be solved with either Dirichlet or Neumann type boundary conditions depending on whether the grid points on the boundary of the surface are to remain fixed or are allowed to move during the construction of the grid. Imposing boundary orthogonality is the most common case where grid points are allowed to move along the boundary. However it is orthogonality of the grid on the surface and not in the parametric region that is desired. Thus the appropriate boundary conditions must be imposed on the elliptic system in the parametric region so that the grid is orthogonal on the surface. Two grid lines will be orthogonal on the surface if the inner product $\vec{r}_\xi \cdot \vec{r}_\eta = 0$. This can be expanded using the chain rule to give the following equation

$$\overline{g}_{11} u_\xi u_\eta + \overline{g}_{22} v_\xi v_\eta + \overline{g}_{12}(u_\xi v_\eta + u_\eta v_\xi) = 0. \tag{3}$$

The orthogonality condition can be formulated as derivative boundary conditions for the above elliptic system. If the boundary segments $u = 0$ and $u = 1$ are considered, then the orthogonality condition reduces to

$$\overline{g}_{22} v_\xi + \overline{g}_{12} u_\xi = 0. \tag{4}$$

Similarly, for the segments $v = 0$ and $v = 1$, orthogonality is imposed by the equation

$$\overline{g}_{11} u_\eta + \overline{g}_{12} v_\eta = 0. \tag{5}$$

Thus the elliptic system together with the given values of u and v on the boundary and the orthogonality boundary conditions form a mixed boundary value problem. Equation (4) determines the values of v along the boundary where $u = 0$ or $u = 1$, and Equation (5) determines the values of u where $v = 0$ or $v = 1$.

The control functions P and Q must be selected so that the grid has the required distribution of grid points in the computational field. Taking $P = Q = 0$ tends to generate a grid with a uniform spacing which is seldom desired. In most cases there is a need to concentrate points in some area of the surface such as along certain boundary contours. There are two methods of computing the control functions. The first is to compute P and Q from some initial grid. All the derivative terms in the elliptic system can be computed from given grid point values leaving two unknowns, P and Q, which can be determined from the two equations. Now these control function values are smoothed so that the final elliptic grid will be smoother and generally more orthogonal than the initial grid. The control functions can be computed from an initial grid only if the Jacobian of the transformation from computational to parametric variables is nonvanishing. Since this may not always be the case for interpolated grids, there is also the option of computing the control functions from the boundary distribution. The appropriate values for the control functions on the boundary can be derived by assuming the grid lines are orthogonal at the boundary and the spacing normal to the boundary is uniform. The development basically follows that for two dimensional plane regions as found in the book by Thompson et. al. (1985). The formulas for P and Q in the surface Equations (1) and (2) are given below.

$$P = -\frac{s_{\xi\xi}}{s_\xi} + s_\xi \mathcal{K}_1 \tag{6}$$

$$Q = -\frac{s_{\eta\eta}}{s_\eta} + s_\eta \mathcal{K}_2 \tag{7}$$

The variable s is the arc length parameter along the boundary of the surface and the variables $\mathcal{K}_1$ and $\mathcal{K}_2$ denote the curvature of the boundary curves $\xi = constant$ and $\eta = constant$,

respectively. The curvatures are given by

$$\mathcal{K}_2 = \sqrt{\frac{G}{g_{22}^3}}\Upsilon_{22}^1$$
$$\mathcal{K}_1 = \sqrt{\frac{G}{g_{11}^3}}\Upsilon_{11}^2$$

where the Christoffel symbols Υ_{ij}^k in the previous formulas are given by

$$\Upsilon_{11}^2 = \frac{1}{2G}\left[g_{11}(2\frac{\partial g_{12}}{\partial \xi} - \frac{\partial g_{11}}{\partial \eta}) - g_{12}\frac{\partial g_{11}}{\partial \xi}\right]$$
$$\Upsilon_{22}^1 = \frac{1}{2G}\left[g_{22}(2\frac{\partial g_{12}}{\partial \eta} - \frac{\partial g_{22}}{\partial \xi}) - g_{12}\frac{\partial g_{22}}{\partial \eta}\right]$$

and

$$G = g_{11}g_{22} - (g_{12})^2.$$

Not all of the terms in Equations (6) and (7) for P and Q can be evaluated on the boundary. It is clear that the arc length derivatives for P can be evaluated on the $\eta = 0$ and $\eta = 1$ boundary curves and interpolated in the interior of the computational region, while the arc length derivatives for Q can be evaluated on $\xi = 0$ and $\xi = 1$ and interpolated in the interior. In a similar manner, the curvature in each control function can be computed on opposite boundaries and interpolated. Note that the curvatures cannot be computed solely from information on the boundary curves. In this manner all of the arc length derivatives and curvatures, and therefore P and Q, can be computed throughout the computational region. This second control option can be used to determine the distribution of grid points on a surface even when an initial grid is of such poor quality in the interior of the surface that it cannot be used itself to generate appropriate control functions.

In the above discussion it was shown how derivative boundary conditions can be used to impose orthogonality. In a numerical algorithm, this requires moving the grid points along the boundary of the surface. Orthogonality can also be imposed by adjusting the control functions near the boundary and keeping the boundary points themselves fixed. Since there are two control functions, not only can the angle at the boundary be chosen, but the distance to the first grid line off of the boundary curve can also be chosen. If it assumed that the grid lines are orthogonal, the values of P and Q can be determined from the elliptic system (1) and (2) yielding the expressions

$$\begin{aligned}
P &= J^2\frac{\overline{g}_{11}\Delta_2 u\ u_\xi + \overline{g}_{22}\Delta_2 v\ v_\xi + \overline{g}_{12}(\Delta_2 u\ v_\xi + \Delta_2 v\ u_\xi}{g_{11}g_{22}}) \\
&- \frac{\overline{g}_{11}u_{\eta\eta}u_\xi + \overline{g}_{22}v_{\eta\eta}v_\xi + \overline{g}_{12}(u_{\eta\eta}v_\xi + v_{\eta\eta}u_\xi}{g_{22}}) \\
&- \frac{\overline{g}_{11}u_{\xi\xi}u_\xi + \overline{g}_{22}v_{\xi\xi}v_\xi + \overline{g}_{12}(u_{\xi\xi}v_\xi + v_{\xi\xi}u_\xi}{g_{11}}) \\
Q &= J^2\frac{\overline{g}_{11}\Delta_2 u\ u_\eta + \overline{g}_{22}\Delta_2 v\ v_\eta + \overline{g}_{12}(\Delta_2 u\ v_\eta + \Delta_2 v\ u_\eta}{g_{11}g_{22}}) \\
&- \frac{\overline{g}_{11}u_{\eta\eta}u_\eta + \overline{g}_{22}v_{\eta\eta}v_\eta + \overline{g}_{12}(u_{\eta\eta}v_\eta + v_{\eta\eta}u_\eta}{g_{22}}) \\
&- \frac{\overline{g}_{11}u_{\xi\xi}u_\eta + \overline{g}_{22}v_{\xi\xi}v_\eta + \overline{g}_{12}(u_{\xi\xi}v_\eta + v_{\xi\xi}u_\eta}{g_{11}})
\end{aligned}$$

Along a boundary component where a specified grid spacing is desired, that distance is used for the value of g_{11} or g_{22}. Some of the other derivatives in the equations for P and Q can be computed from the fixed values of u and v on the boundary. Those derivatives that must be computed using interior values are updated during the iterative solution of the elliptic system. At each iteration, these boundary control functions are smoothly blended with the interior control functions to generate a smooth grid that is orthogonal and has a specified spacing along assigned boundary components.

EXAMPLES

Two computational examples will be presented to demonstrate the utility of the method in grid generation. In both cases the surface is defined as a NURBS (NonUniform Rational B-Spline) surface. The elliptic system is solved using a finite difference discretization and an SOR iterative method. In some cases, the SOR method does not converge. In those cases, an acceptable grid can often be generated by setting the right hand sides of Equations (1) and (2) to zero. Of course, curvature information about the surface is lost since all of the second order derivatives with respect to the parametric variables appear on the right hand side of the equations. In both of the examples the grids are very coarse so that the grid properties can be clearly seen from the plots.

The first example is a simple surface with a complex parameterization. Two grids are shown in Figure 1. One grid is constructed by transfinite interpolation in the parametric region. The other grid is generated by smoothing the first grid with the elliptic system. The second grid is certainly smoother and more uniform. The control functions were taken to be zero and the boundary points were allowed to move so that the grid is orthogonal at the boundary. Some nonuniformity is noted in the grid, but these are due to small changes in the curvature of the surface and not to the parameterization. At this point, it is not known if these effects are inherent in the elliptic system or are due to truncation error in the discretization.

The second example is a more realistic configuration. A space shuttle type geometry is depicted with an interpolated and a smoothed grid in Figure 2. Due to the large curvature of the surface, convergence problems were encountered in the numerical solution of the elliptic system. Therefore the right hand sides of the elliptic system in Equations (1) and (2) were set to zero. Although there is not a great deal of difference in the two grids, the elliptic grid does result in a more uniform distribution especially in front of the wing where the grid constructed by interpolation has a sparser distribution of points.

CONCLUSIONS AND RECOMMENDATIONS

The elliptic equations for surface grids have been presented along with the control functions and orthogonality techniques which allow a great deal of freedom in designing high quality grids. This capability would be most useful when the parameterization of the surface is such that interpolation of parametric values does not give a satisfactory grid because of a poorly parameterized surface. Since this situation arises frequently when surfaces are defined by CAD packages, the capability to smooth and improve surface grids is essential in any state-of-the-art grid generation code. The method described in this report works well on surfaces which have a smooth parametric representation with moderate values for the parametric derivatives. If there are large variations in the parametric derivatives, then the elliptic system becomes difficult to solve and the standard iterative methods, such as SOR, will not converge. However, this is precisely the case when the elliptic grid generation

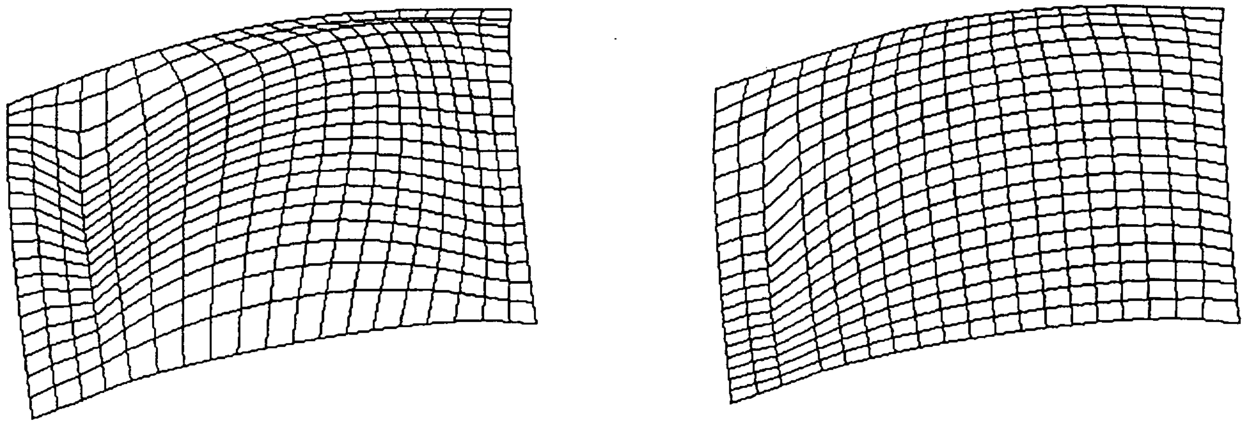

Figure 1: Initial Grid (left) and Elliptic Grid (right) for a Surface

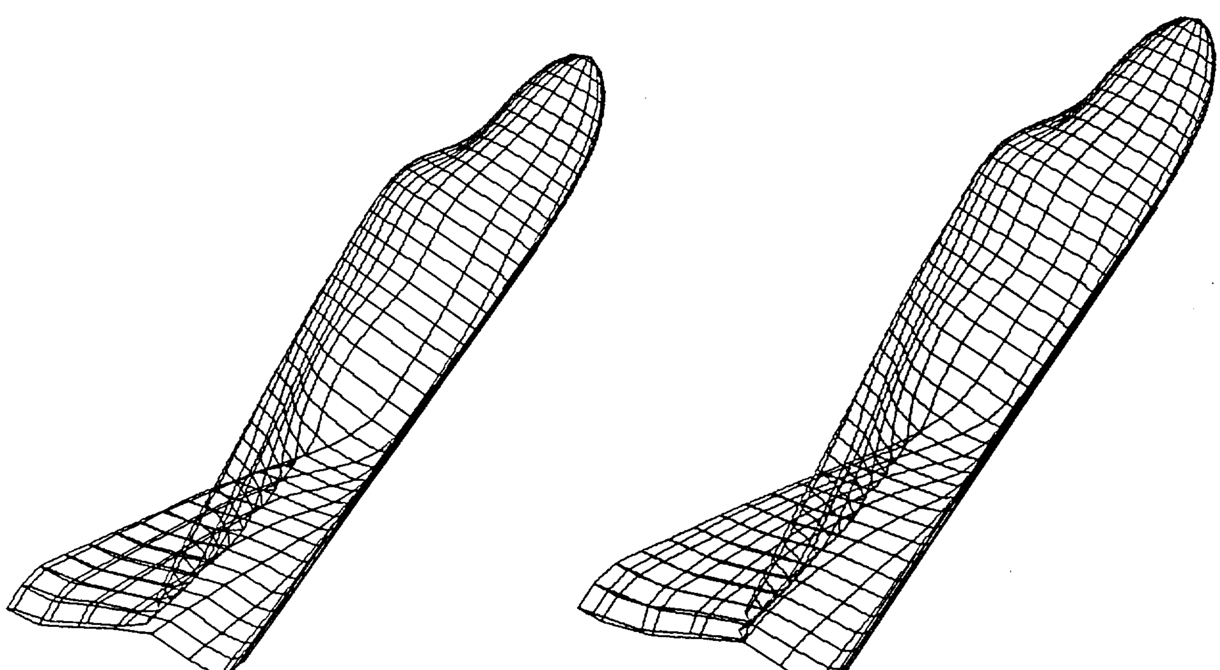

Figure 2: Initial Grid (left) and Elliptic Grid (right) for a Shuttle Configuration

methods can be most useful. Thus there is a need for more study on algorithms to solve nonlinear elliptic systems such as those encountered in this report.

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